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Variational iteration method for solving a generalized pantograph equation

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ABSTRACT

The variational iteration method is applied to solve the generalized pantograph equation. This technique provides a sequence of functions which converges to the exact solution of the problem and is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. Employing this technique, it is possible to find the exact solution or an approximate solution of the problem. Some examples are given to demonstrate the validity and applicability of the method and a comparison is made with existing results.

1. Introduction

Pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of Ockendon and Tayler [1] on the collection of current by the pantograph head of an electric locomotive. The pantograph equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. For some applications of this equation we refer the interested reader to [1–3,12,39]. Properties of the analytic solution of this equation as well as numerical methods have been studied by several authors [4–10,12,39].

In [6] the authors applied the Taylor method to approximate solution of the nonhomogenous multi-pantograph equation with variable coefficients, which is extended of the multi-pantograph equation given in [8,11],

$$u'(t) = \beta u(t) + \sum_{i=1}^{\ell} \mu_i(t)u(q_i t) + f(t), \quad t \geq 0, \tag{1}$$

under the condition $u(0) = \gamma$, where $\beta, \gamma \in \mathcal{C}$; $\mu_i(t)$ and $f(t)$ are analytical functions; $0 < q_i < 1$. Also the properties of the analytic and the numerical solutions of (1) with $f(t) = 0$ and $\mu_i(t) = \mu_i$, are discussed in [8]. The authors of [7,12] suggested a numerical method for solution of the generalized pantograph equations with linear functional argument

$$u^{(m)}(t) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(t)u^{(k)}(\alpha_j t + \beta_j) + f(t), \tag{2}$$

with the initial conditions

$$\sum_{k=0}^{m-1} c_{ik}u^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1, \tag{3}$$

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where $P_{jk}(t)$ and $f(t)$ are analytical functions; c_{ik}, λ_i, α_j and β_j are real or complex constants. In this paper we consider the following two functional differential equations

Problem 1.

$$\begin{cases} u'(t) = \beta u(t) + f\left(t, u(t), u(\alpha_1(t)), u(\alpha_2(t)), \ldots, u(\alpha_\ell(t))\right), \\ u(0) = u_0. \end{cases} \tag{4}$$

Problem 2.

$$\begin{cases} u^{(m)}(t) = f\left(t, u(t), u(\alpha_1(t)), u(\alpha_2(t)), \ldots, u(\alpha_\ell(t))\right), \\ \displaystyle\sum_{k=0}^{m-1} c_{ik} u^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1. \end{cases} \tag{5}$$

Here f and $\alpha_i(t)$, $i = 1, 2, \ldots, \ell$, are analytical functions; c_{ik}, λ_i and β are real or complex constants. A classical case that is the subject of a lot of papers is the following:

$$\alpha_i(t) = t - \tau_i, \quad i = 1, 2, \ldots, \ell,$$

where τ_i, $i = 1, 2, \ldots, \ell$ is a positive constant [13,14]. Another interesting case [15,16], which is used in the pantograph equation, is

$$\alpha_i(t) = q_i t, \quad i = 1, 2, \ldots, \ell,$$

where $0 < q_\ell < q_{\ell-1} < \cdots < q_1 < 1$.

In this work, we employ the variational iteration method (shortly VIM) for solving Problems 1 and 2. The He's VIM gives several successive approximations through using the iteration of the correction functional. This method was proposed by the Chinese researcher Ji-Huan He [17–19] as a modification of a general Lagrange multiplier method [20]. The VIM does not require specific transformations for nonlinear terms as required by some existing techniques. An elementary introduction of VIM is given in [21]. The main concepts in variational iteration method, such as general Lagrange multiplier, restricted variation, correction functional are explained heuristically. Subsequently, the solution procedure is systematically addressed, in particular, for nonlinear oscillators. The VIM plays an important role in recent researches for solving various kinds of problems (see for example [22–26] and the references therein). The variational iteration method is used in [27] to solve the Lane–Emden differential equation. VIM is employed in [28] to solve the Klein–Gordon partial differential equation. This method is applied in [29] to a system of integral equations arising in biology and describing biological species living together. The He's variational iteration technique is proposed in [30] to solve the Cauchy-reaction-diffusion equation. VIM is developed to find the solution of a model arising in a biological population model [31]. The variational iteration method is used in [22,40] to solve some parabolic inverse problems. This method is employed in [41] to solve the wave equation subject to an integral conservation condition. Some problems in calculus of variations are solved using the variational iteration method [42]. VIM is investigated in [43] to solve parabolic integro-differential equations arising in heat conduction in materials with memory. This technique is applied in [44] to find the solution of nonlinear mixed Volterra–Fredholm integral equations. The variational iteration method is improved in [45] to solve a system of differential equations. Four important classes of variational problems are solved in [46] using VIM. For a relatively comprehensive survey on the method and its applications, the readers are referred to the review article [32] and monograph [33].

A well-known analytic approach in the literature for solving differential equations is Adomian decomposition method. But the main disadvantage of the Adomian is that the solution procedure for calculation of Adomian polynomials is complex and difficult as pointed by many researchers. A complete comparison between the Adomian decomposition method and the VIM is available on [34]. Also some new developments and new interpretations of VIM are available on [21]. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. This technique solves the problem without any need to discretization of the variables [35]. Therefore, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. This scheme provides the solution of the problem in a closed form. Also this procedure is useful for finding an accurate approximation of the exact solution.

The organization of this paper is as follows:

In Section 2, we apply the VIM on the Problems 1 and 2. To present a clear overview of method, in Section 3 we give several examples with analytical solutions, and solve them using the variational iteration technique and a comparison is made with existing results. A brief conclusion is given in Section 4.

2. The application of VIM

In this section the application of the VIM is discussed for solving Problems 1 and 2. As stated before, VIM is based on the general Lagranges multiplier method. To illustrate the basic concept of VIM, we consider the following general non-linear differential equation:

$$Lu + Nu = g(t), \tag{6}$$

where L is a linear operator, N is a non-linear operator, and $g(t)$ is a known analytical function. According to VIM we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\{Lu_n(s) + N(\tilde{u}_n(s)) - g(s)\}\mathrm{d}s, \quad n \geq 0, \tag{7}$$

where λ is a general Lagrangian multiplier [20] which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n = 0$, [17–19]. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrangian multiplier and any selective function u_0. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

First we consider Problem 1, according to the VIM, we consider the correction functional in the following form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left\{u_n'(s) - \beta u_n(s) - f\left(s, \tilde{u}_n(s), \tilde{u}_n(\alpha_1(s)), \tilde{u}_n(\alpha_2(s)), \ldots, \tilde{u}_n(\alpha_\ell(s))\right)\right\}\mathrm{d}s. \tag{8}$$

To find the optimal value of λ we have

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta\int_0^t \lambda(s)\left\{u_n'(s) - \beta u_n(s)\right\},$$

that results

$$\delta u_{n+1}(t) = \delta u_n(t) + \lambda\delta u_n(s)|_{s=t} - \int_0^t \{\lambda'(s) + \beta\lambda(s)\}\delta u_n(s)\mathrm{d}s = 0.$$

Thus we have the following stationary conditions:

$$\delta u_n : 1 + \lambda(t) = 0,$$
$$\delta u_n : \lambda'(s) + \beta\lambda(s) = 0.$$

Therefore, the Lagrange multiplier can be readily identified as $\lambda = -\mathrm{e}^{-\beta(s-t)}$. As a result, we obtain the following iteration formula:

$$u_{n+1}(t) = u_n(t) - \int_0^t \mathrm{e}^{-\beta(s-t)}\left\{u_n'(s) - \beta u_n(s) - f\left(s, u_n(s), u_n(\alpha_1(s)), u_n(\alpha_2(s)), \ldots, u_n(\alpha_\ell(s))\right)\right\}\mathrm{d}s. \tag{9}$$

Similarly, the correct functionals for Problem 2 can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left\{u_n^{(m)}(s) - f\left(s, \tilde{u}_n(s), \tilde{u}_n(\alpha_1(s)), \tilde{u}_n(\alpha_2(s)), \ldots, \tilde{u}_n(\alpha_\ell(s))\right)\right\}\mathrm{d}s. \tag{10}$$

By the same manipulation as in the previous part the Lagrange multiplier is [36]

$$\lambda = (-1)^m\frac{(s-t)^{m-1}}{(m-1)!},$$

therefore, we have the following iteration formula:

$$u_{n+1}(t) = u_n(t) + (-1)^m\int_0^t \frac{(s-t)^{m-1}}{(m-1)!}\left\{u_n^{(m)}(s) - f\left(s, u_n(s), u_n(\alpha_1(s)), u_n(\alpha_2(s)), \ldots, u_n(\alpha_\ell(s))\right)\right\}\mathrm{d}s. \tag{11}$$

3. Numerical examples

To incorporate our discussion above, in this section, we will apply the VIM to solve the pantograph equation. Results obtained by the VIM are compared with the exact solution of each example and are found to be in good agreement with each other. Here, we used the Maple package to calculate the terms of the sequence obtained from the VIM.

Example 1. Consider the following equation [6,7,37]

$$u'(t) = \frac{1}{2}u(t) + \frac{1}{2}\mathrm{e}^{t/2}u\left(\frac{t}{2}\right), \qquad u(0) = 1, \quad 0 \leq t \leq 1, \tag{12}$$

which has the exact solution $u(t) = \mathrm{e}^t$. For this example we can write iteration formula (9) as

$$u_{n+1}(t) = u_n(t) - \int_0^t \mathrm{e}^{-\frac{1}{2}(s-t)}\left\{u_n'(s) - \frac{1}{2}u_n(s) - \frac{1}{2}\mathrm{e}^{s/2}u_n\left(\frac{s}{2}\right)\right\}\mathrm{d}s. \tag{13}$$

Table 1
Comparison of the absolute errors for Example 1.

t	Adomian method with 13 terms [37]	Taylor method [7] with $N = 16$	The present method $n = 6$	$n = 7$	$n = 8$
0.2	0.00	2.22×10^{-16}	2.25×10^{-17}	2.20×10^{-21}	9.60×10^{-26}
0.4	2.22×10^{-16}	2.22×10^{-16}	3.43×10^{-15}	6.75×10^{-19}	5.88×10^{-23}
0.6	2.22×10^{-16}	2.22×10^{-16}	6.99×10^{-14}	2.06×10^{-17}	2.70×10^{-21}
0.8	1.33×10^{-15}	0.00	6.28×10^{-13}	2.46×10^{-16}	4.32×10^{-20}
1.0	4.88×10^{-15}	2.22×10^{-15}	3.55×10^{-12}	1.75×10^{-15}	3.85×10^{-19}

Table 2
Comparison of the absolute errors for $q = 0.2$ for Example 2.

t	Collocation method $m = 2$ [38]	Taylor method [7] with $N = 16$	The present method $n = 3$	$n = 4$	$n = 5$
2^{-1}	2.71×10^{-5}	7.77×10^{-16}	1.47×10^{-10}	2.40×10^{-15}	6.51×10^{-21}
2^{-2}	1.08×10^{-6}	1.11×10^{-16}	9.79×10^{-12}	7.91×10^{-17}	1.05×10^{-22}
2^{-3}	3.81×10^{-8}	2.22×10^{-16}	6.31×10^{-13}	2.53×10^{-18}	1.00×10^{-24}
2^{-4}	1.26×10^{-9}	0.00	4.00×10^{-14}	8.03×10^{-20}	0.00
2^{-5}	4.09×10^{-11}	1.11×10^{-16}	2.52×10^{-15}	2.52×10^{-21}	0.00
2^{-6}	1.20×10^{-12}	0.00	1.58×10^{-12}	7.00×10^{-23}	0.00

Let us start with an initial approximation $u_0(t) = 1$ and use iteration formula (13). We can obtain directly the other components as

$$\begin{aligned}
u_1(t) &= 1 + t + O(t^2),\\
u_2(t) &= 1 + t + \frac{t^2}{2} + O(t^3),\\
u_3(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + O(t^4),\\
&\vdots\\
u_n(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + O(t^{n+1}),
\end{aligned}$$

that gives the exact solution by $u(t) = \lim_{n\to\infty} u_n(t) = e^t$. In Table 1 we compare the absolute error of VIM with $n = 6, 7$ and $n = 8$ together with the Adomian decomposition method [37] and Taylor method [7].

Example 2. In this example, we consider pantograph equation of first order

$$u'(t) = -u(t) + \frac{q}{2}u(qt) - \frac{q}{2}e^{-qt}, \quad u(0) = 1, \tag{14}$$

which has the exact solution $u(t) = e^{-t}$. Note that $q = 1$ is not a pantograph equation, is a linear differential equation.

The iteration formula (9) for this example is

$$u_{n+1}(t) = u_n(t) - \int_0^t e^{s-t}\left\{u_n'(s) + u_n(s) - \frac{q}{2}u_n(qs) + \frac{q}{2}e^{-qs}\right\} ds. \tag{15}$$

Again we start with an initial approximation $u_0(t) = 1$ and substitute this equation into Eq. (15). Table 2 compares the results of the present method and the collocation method of [38] and Taylor method of [7].

Example 3. Consider the pantograph equation of first order with variable coefficients

$$\begin{cases} u'(t) = -u(t) - e^{\frac{-t}{2}}\sin\left(\frac{t}{2}\right)u\left(\frac{t}{2}\right) - 2e^{-\frac{3t}{4}}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{4}\right)u\left(\frac{t}{4}\right), \quad 0 \le t \le 1,\\ u(0) = 1. \end{cases} \tag{16}$$

To solve this equation using the VIM, based on the iteration formula (9) we get

$$u_{n+1}(t) = u_n(t) - \int_0^t e^{s-t}\left\{u_n'(s) + u_n(s) + e^{\frac{-s}{2}}\sin\left(\frac{s}{2}\right)u_n\left(\frac{s}{2}\right) + 2e^{-\frac{3s}{4}}\cos\left(\frac{s}{2}\right)\sin\left(\frac{s}{4}\right)u_n\left(\frac{s}{4}\right)\right\} ds. \tag{17}$$

Table 3
Comparison of the absolute errors for Example 3.

t	Taylor method [6] with $N = 9$	Present method $n = 3$	$n = 4$	$n = 5$
0.0	0.0	0.0	0.0	0.0
0.1	3.0×10^{-11}	3.6×10^{-13}	1.6×10^{-17}	2.7×10^{-17}
0.2	1.3×10^{-9}	4.2×10^{-11}	7.5×10^{-16}	7.5×10^{-17}
0.3	2.0×10^{-8}	6.6×10^{-10}	2.5×10^{-14}	6.9×10^{-17}
0.4	1.4×10^{-7}	4.5×10^{-9}	3.1×10^{-13}	8.2×10^{-17}
0.5	6.3×10^{-7}	1.9×10^{-8}	2.0×10^{-12}	5.9×10^{-17}
0.6	2.0×10^{-6}	6.3×10^{-8}	9.7×10^{-12}	3.2×10^{-16}
0.7	5.4×10^{-6}	1.6×10^{-7}	3.5×10^{-11}	1.5×10^{-15}
0.8	1.2×10^{-5}	3.8×10^{-7}	1.0×10^{-10}	5.9×10^{-15}
0.9	2.3×10^{-5}	7.9×10^{-7}	2.7×10^{-10}	1.9×10^{-14}
1.0	4.0×10^{-5}	1.5×10^{-6}	6.3×10^{-10}	5.6×10^{-14}

We start with initial approximation $u_0(t) = 1$, the other terms of the sequence $u_n(t)$ are computed easily by substituting this equation into Eq. (17) we have

$$u_1(t) = 1 - t + \frac{5}{24}t^3 + \cdots,$$
$$u_2(t) = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{539}{15360}t^5 + \cdots,$$
$$u_3(t) = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{1}{30}t^5 - \frac{1051249}{660602880}t^7 \cdots,$$
$$u_4(t) = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{1}{30}t^5 - \frac{1}{630}t^7 + \frac{1}{2520}t^8 + 0.0000440899t^9 + \cdots.$$

Note that the exact solution of this problem is [6]

$$u(t) = e^{-t}\cos t = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{1}{30}t^5 - \frac{1}{630}t^7 + \frac{1}{2520}t^8 + 0.0000440917t^9 + \cdots.$$

We see that the approximation solutions obtained by VIM have good agreement with exact solution of this problem. In Table 3 the absolute errors of the present method for $n = 3, 4, 5$ and Taylor method of [6] with 9 terms are compared.

Example 4. Consider the pantograph equation of second order [7]

$$\begin{cases} u''(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2, & 0 \le t \le 1, \\ u(0) = 0, \qquad u'(0) = 0, \end{cases} \tag{18}$$

which has the exact solution $u(t) = t^2$. The iteration formula (11) for this example is

$$u_{n+1}(t) = u_n(t) + \int_0^t (s - t)\left\{u_n''(s) - \frac{3}{4}u_n(s) - u_n\left(\frac{s}{2}\right) + s^2 - 2\right\} ds. \tag{19}$$

Starting with an initial approximation $u_0(t) = 0$ and substituting this equation into Eq. (19) yield

$$u_1(t) = t^2 - \frac{1}{12}t^4,$$
$$u_2(t) = t^2 - \frac{13}{5760}t^6,$$
$$u_3(t) = t^2 - \frac{91}{2949120}t^8,$$
$$\vdots$$
$$u_n(t) = t^2 - \text{small term},$$

that gives the exact solution by $u(t) = \lim_{n\to\infty} u_n(t) = t^2$.

Example 5. Consider the pantograph equation of third order [7,37]

$$\begin{cases} u'''(t) = -u(t) - u(t - 0.3) + e^{-t+0.3}, & 0 \le t \le 1, \\ u(0) = 1, \qquad u'(0) = -1, \qquad u''(0) = 1. \end{cases} \tag{20}$$

Table 4
Comparison of the absolute errors for Example 5.

t	Adomian method with 6 terms [37]	Taylor method [7] with $N = 17$	The present method		
			$n = 6$	$n = 8$	$n = 10$
0.0	8.52×10^{-14}	0.00	0.00	0.00	0.00
0.2	3.83×10^{-14}	0.00	1.59×10^{-15}	4.69×10^{-19}	1.50×10^{-22}
0.4	1.68×10^{-13}	2.22×10^{-16}	7.80×10^{-15}	2.29×10^{-18}	7.35×10^{-22}
0.6	6.00×10^{-14}	1.11×10^{-16}	1.65×10^{-14}	4.89×10^{-18}	1.57×10^{-21}
0.8	6.66×10^{-15}	0.00	2.07×10^{-14}	6.33×10^{-18}	2.07×10^{-21}
1.0	4.57×10^{-14}	5.55×10^{-17}	7.55×10^{-15}	2.95×10^{-18}	1.08×10^{-21}

The exact solution of this problem is $u(t) = \mathrm{e}^{-t}$. If we want to solve this equation by means of VIM, using (11) we have

$$u_{n+1}(t) = u_n(t) - \int_0^t \frac{(s-t)^2}{2!} \left\{u_n'''(s) + u_n(s) + u_n(s-0.3) - \mathrm{e}^{-s+0.3}\right\} \mathrm{d}s. \tag{21}$$

Choosing $u_0(t) = A + Bt + Ct^2$, the other terms of the sequence $u_n(t)$ are computed easily using symbolic computations. Imposing the initial conditions, $u(0) = 1$, $u'(0) = -1$ and $u''(0) = 1$, into $u_6(t)$ we have $A = 1$, $B = -1$, $C = 0.5$. Also By imposing the initial conditions into $u_8(t)$ and $u_{10}(t)$ we get $A = 1$, $B = -1$, $C = 0.5$. In Table 4 we compare the absolute errors of the present method for $n = 6, 8, 10$ and the Adomian decomposition method described in [37] with six terms and Taylor method developed in [7] with 17 terms.

All the examples show that the results of the VIM are in excellent agreement with those of other methods. They showed that VIM with the fewest number of iterations can converge to correct results.

The technique developed in this paper can be used to solve the second Painleve equation investigated in [47].

4. Conclusion

In this work, He's variational iteration method has been successfully applied to the generalized pantograph equation. Comparing with other methods, the results of numerical examples demonstrate that this method is more accurate than the stated existing methods and few iterations are enough to obtain highly accurate solution. This technique produces the terms of a sequence using the iteration of the correction functional which converges to the exact solution rapidly. Also the variational iteration technique provides the solution of the problem without calculating Adomian's polynomials which is an important advantage over the Adomian decomposition method. Furthermore this technique solves the problem without any need to discretization of the variables. Therefore, it is not affected by computation round off errors. Overall, the reliability of the method and the reduction of the size of computational domain give this method wide applications.

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